Appendix A: Mathematical Formulas

Exponentials and Logarithms

\[ e^{\ln x} = x, \quad \ln(e) = 1, \quad \ln(e^a) = a, \quad \ln(x^a) = a \ln x. \]
\[ e^a e^b = e^{a+b}, \quad \ln(xy) = \ln x + \ln y, \quad \ln(x/y) = \ln x - \ln y. \]

Solid Bodies

Volume of sphere radius \( R \): \( \frac{4\pi R^3}{3} \).
Surface area of sphere radius \( R \): \( 4\pi R^2 \).
Volume of cone, base \( R \), height \( h \): \( \frac{\pi h R^2}{3} \).

Vectors

A vector can be expressed in terms of 3 orthogonal, normalised basis vectors which are commonly denoted \( \mathbf{i}, \mathbf{j} \) and \( \mathbf{k} \) (or \( \hat{x}, \hat{y} \) and \( \hat{z} \)): \( \mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k} \), or represented by the number triplet of its components: \( \mathbf{a} = (a_x, a_y, a_z) \).

The modulus of \( \mathbf{a} \) is \( |\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} = \sqrt{\mathbf{a} \cdot \mathbf{a}} \).

Scalar product: if \( \theta \) is the angle between two vectors \( \mathbf{a} \) and \( \mathbf{b} \), the scalar product is
\[ \mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z = |\mathbf{a}||\mathbf{b}| \cos \theta. \]

Vector product: the direction of the vector product is normal to both \( \mathbf{a} \) and \( \mathbf{b} \) and has magnitude \( |\mathbf{a}||\mathbf{b}| \sin \theta \).

Differentiation and Integration

\[
\frac{d}{dx}(x^n) = nx^{n-1} \quad \int x^n dx = \frac{1}{n+1}x^{n+1} + c
\]
\[
\frac{d}{dx}(e^{ax}) = ae^{ax} \quad \int e^{ax} dx = \frac{1}{a}e^{ax}
\]
\[
\frac{d}{dx}(\ln x) = \frac{1}{x} \quad \int \frac{1}{x} dx = \ln |x| + c
\]
\[
\frac{d}{dx}(1/x) = -\frac{1}{x^2} \quad \int \frac{1}{x^2} dx = -\frac{1}{x} + c
\]
\[
\frac{d}{dx} \sin x = \cos x \quad \int \cos x dx = \sin x + c
\]
\[
\frac{d}{dx} \cos x = -\sin x \quad \int \sin x dx = -\cos x + c
\]
\[
\frac{d}{dx} \tan x = \sec^2 x \quad \int \sec^2 x \, dx = \tan x + c
\]
\[
\frac{d}{dx} (a^x) = a^x \ln a \quad \int a^x \, dx = \frac{1}{\ln a} a^x + c
\]
In the above \(a\) and \(c\) are constants with \(c\) the constant of integration.

- **Product rule for differentiation**
  \[
  \frac{d}{dx} (f_1 f_2) = f_1 \frac{df_2}{dx} + f_2 \frac{df_1}{dx}
  \]

- **Quotient rule for differentiation**
  \[
  \frac{d}{dx} \left( \frac{f_1}{f_2} \right) = \frac{f_2 (d f_1 / dx) - f_1 (d f_2 / dx)}{f_2^2}
  \]

- **Integration by parts**
  \[
  \int f(x) \left( \frac{dg(x)}{dx} \right) \, dx = f(x) g(x) - \int g(x) \left( \frac{df(x)}{dx} \right) \, dx
  \]

### Complex numbers

In terms of its modulus \(r\) and angle \(\theta\):

\[
z = x + jy = re^{j\theta} = r(\cos \theta + j \sin \theta).
\]

Under complex conjugation \(j \rightarrow -j\):

\[
z^* = x - jy = r e^{-j\theta} = r(\cos \theta - j \sin \theta).
\]

The modulus squared of \(z\) is the real number \(|z|^2 = z^* z = x^2 + y^2 = r^2\).

### Trigonometric functions

\[
\sin^2 x + \cos^2 x = 1
\]
\[
\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y
\]
\[
\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y
\]

The following may be derived easily from the above:

\[
\sin x \cos y = \frac{1}{2}(\sin(x + y) + \sin(x - y))
\]
\[
\cos x \cos y = \frac{1}{2}(\cos(x + y) + \cos(x - y))
\]
\[
\sin x \sin y = \frac{1}{2} (\cos(x - y) - \cos(x + y))
\]
\[
\sin x \pm \sin y = 2 \sin \frac{1}{2}(x \pm y) \cos \frac{1}{2}(x \mp y)
\]
\[
\cos x + \cos y = 2 \cos \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y)
\]
\[
\cos x - \cos y = -2 \sin \frac{1}{2}(x + y) \sin \frac{1}{2}(x - y)
\]
\[
\sin 2x = 2 \sin x \cos x
\]
\[
\cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x
\]
\[
\cos \theta = \frac{1}{2} (e^{j\theta} + e^{-j\theta})
\]
\[
\sin \theta = \frac{1}{2j} (e^{j\theta} - e^{-j\theta})
\]

**Series**

- **Geometric series:**
  \[
  1 + x + x^2 + \ldots + x^{n-1} = \frac{1 - x^n}{1 - x} \rightarrow \frac{1}{1 - x}
  \]
  \(-1 < x < 1\)

- **Binomial expansion:**
  \[
  (1 + x)^n = 1 + nx + \frac{n(n - 1)}{2!} x^2 + \frac{n(n - 1)(n - 2)}{3!} x^3 + \ldots
  \]
  If \( n \) is a positive integer the series terminates after \( n + 1 \) terms, otherwise it is an infinite series.

- **Taylor Series about the point \( x = x_0 \):**
  \[
  f(x) = f(x_0) + (x - x_0) \left( \frac{df}{dx} \right)_{x_0} + \frac{1}{2!} (x - x_0)^2 \left( \frac{d^2f}{dx^2} \right)_{x_0} + \ldots
  \]
  The following series are frequently encountered:
  \[
  e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots
  \]
  \[
  \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots
  \]
  \[
  \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots
  \]
  \[
  \ln(1 \pm x) = \pm x - \frac{x^2}{2} \pm \frac{x^3}{3} - \ldots
  \]
\((1 \mp x)^{-1} = 1 \pm x + x^2 \pm x^3 + \ldots \text{for } -1 < x < 1.\)

L’Hospital’s rule follows from Taylor’s series: if \(f(0) = g(0) = 0,\)
\[
\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{df/dx}{dg/dx}.
\]
\[
\lim_{x \to 0} \frac{\sin x}{x} = 1.
\]

**Differential Equations**

Important differential equations and their solutions:

\[
\frac{dy}{dt} + \lambda y = 0 \quad y = Ae^{-\lambda t}
\]
\[
a \frac{d^2y}{dt^2} + cy = 0 \quad y = B \cos \omega t + C \sin \omega t = A \cos(\omega t + \phi)
\]
with \(\omega = \sqrt{c/a}\)
\[
a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0 \quad y = Ae^{-\gamma t} \cos(\omega t + \phi)
\]
where \(a, b, c, A, B, C\) and \(\phi\) are constants. In the last equation \(\gamma = b/2a\) and \(\omega = \sqrt{c/a - b^2/4a^2}\) and the solution given is correct for \(a, b\) and \(c\) all positive.

**Partial Differentiation**

Given a function \(f\) of more than one variable, the infinitesimal change in \(f\) when the variables are changed infinitesimally is
\[
df(x, y) = \left( \frac{\partial f}{\partial x} \right)_y \, dx + \left( \frac{\partial f}{\partial y} \right)_x \, dy
\]

**Coordinate Systems**

- **Cartesian coordinates:** \((x, y, z)\)
  
The position vector \(\mathbf{r} = xi + yj + zk\)
where \(x, y,\) and \(z\) are the cartesian coordinates of the point.
  
Volume element \(dx \, dy \, dz\)

- **Spherical polar coordinates:** \((r, \theta, \phi)\)
  
\[r = \sqrt{x^2 + y^2 + z^2}; \quad \theta = \arctan(\sqrt{x^2 + y^2}/z); \quad \phi = \arctan(y/x);\]
\[x = r \sin \theta \cos \phi; \quad y = r \sin \theta \sin \phi; \quad z = r \cos \theta\]
Volume element: $dV = r^2 \sin \theta \, dr \, d\theta \, d\phi$;

Area element on the surface of a sphere: $dA = r^2 \sin \theta \, d\theta \, d\phi$

• **Cylindrical polar coordinates:** $(r, \phi, z)$

Volume element: $dV = r \, dr \, d\phi \, dz$