## 6 Series

Series were first introduced in Section 1 and defined in equation 1.4. They are summations of successive terms, each of which has a structure which depends on the particular series. The series can have an infinite or a finite number of terms. A simple example of a finite series is the **arithmetic series**,

$$\sum_{n=n_1}^{n=n_2} n = \frac{1}{2}\left(n_2(n_2+1) - n_1(n_1+1)\right). \tag{6.1}$$

Each term increases by one and the series begins at $n = n_1$ and ends at $n = n_2$. If $a$ is a constant

$$\sum_{n=n_1}^{n=n_2} an = \frac{a}{2}\left(n_2(n_2+1) - n_1(n_1+1)\right). \tag{6.2}$$

Another example of a finite series is the **geometric series**, in which successive terms equal the previous term multiplied by a constant.

$$\sum_{n=0}^{n=N-1} ab^n = a\frac{(b^N-1)}{(b-1)}, \tag{6.3}$$

where $a$ and $b$ are constants.

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**Problem 6.1** Show that

$$\sum_{n=2}^{n=5} 3x^n = \frac{3x^2}{(x-1)}(x^4-1)$$

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### 6.1 Exponential series

The exponential function given by equation (2.6) can be expressed as a series. We need a series which when differentiated returns the original series, in agreement with to the definition of the exponential function given in equation 2.5. Accordingly, the function $e^x$ can be written

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}.., \tag{6.4}$$

$$= \sum_{n=0}^{n=\infty} \frac{x^n}{n!}, \tag{6.5}$$

where $n$ is an integer, and $n!$ is factorial $n$ given by equation 2.7.

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}.., \tag{6.6}$$

Section 5 gives relationships between sines, cosines and exponential functions with imaginary arguments. These equations can be used to express $\sin x$ and $\cos x$ in terms of exponentials and then rewrite them as series using the above expansions of exponential functions. From equations 5.16 and 5.17,

$$\sin x = \frac{1}{2j}(e^{jx} - e^{-jx}), \tag{6.7}$$

and

$$\cos x = \frac{1}{2}(e^{jx} + e^{-jx}). \tag{6.8}$$

Using equations 6.4 and 6.6 these become

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \quad .... \tag{6.9}$$

and

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \quad ... \tag{6.10}$$

It may not be necessary, but a reminder that in formulae such as 6.9 and 6.10 the angles $x$ are in the natural units of radians may be timely.

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**Problem 6.2** Derive equation 6.9.

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### 6.2 Binomial theorem

The **binomial expansion** is a series representation of the function $(1 + x)$ raised to the power $n$, where $n$ is any real number. The theorem is that

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 ... \tag{6.11}$$

The expansion stops at the $n$'th term after the first (the term whose numerator becomes zero) when $n$ is a positive integer, otherwise the series is infinite.

The proof of the binomial theorem is established by assuming it to be true and differentiating both sides.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(1 + x)^n &= n(1 + x)^{n-1} \\
&= n\left(1 + (n-1)x + \frac{(n-1)(n-2)}{2!}x^2 + ...\right) \\
&= n + n(n-1)x + \frac{n(n-1)(n-2)}{2!}x^2 + ...
\end{aligned}$$

This is equal to the expression obtained by differentiating the right-hand side of equation 6.11, and constitutes a proof of the validity of that equation.

Equation 6.11 is an important equality and especially useful when $x$ is small compared with unity, when the ratio of successive terms decreases rapidly. In that case, the series need only be taken as far as justified by the problem being considered.

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**Example 6.1** The mass $m$ of an object moving at speed $v$ is multiplied by the factor $(1 - v^2/c^2)^{-1/2}$ when special relativity is taken into account. The constant $c$ is the speed of light in free space and can be taken equal to $3 \times 10^8$ m s$^{-1}$. By what percentage is the mass of a rocket travelling at $v = 10^6$ km per hour increased?

*Solution* The ratio $v/c = 9.26 \times 10^{-4}$ to two places of decimals, and $(v/c)^2 = 8.57 \times 10^{-7}$. The binomial expansion of $(1 - v^2/c^2)^{-1/2}$ is

$$(1 - v^2/c^2)^{-1/2} = 1 - \frac{1}{2}x + (-\frac{1}{2})(-\frac{3}{2})\frac{1}{2!}x^2 + \quad ....$$

with $x = (-v/c)^2$. The second term in the above equation has magnitude $4.29 \times 10^{-7}$. The third term is smaller than the second by the same amount and can be neglected, giving $(1 - v^2/c^2)^{-1/2} = 1 + 4.29 \times 10^{-7}$ and the percentage increase in mass close to $4.3 \times 10^{-5}$.

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**Problem 6.3** Show that

$$\frac{1}{(1-x)} = \sum_{n=0}^{n=\infty} x^n,$$

for values of $|x|$ less than unity.

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Series expansions of other functions may be determined in a manner similar to that in which the series expansions of the exponential functions were obtained, namely by finding a series which when differentiated gives the same differential as that of the function.

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**Example 6.2** Show that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \quad ....$$

*Solution*

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(1+x) = \frac{1}{(1+x)}$$

from equation 2.8. The binomial theorem may now be used to expand this, giving

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(1+x) = 1 - x + x^2 - x^3 + x^4 \quad ...$$

But

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \quad ..\right) = 1 - x + x^2 - x^3 + x^4 \quad ...$$

Hence

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \quad ....$$

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**Problem 6.4** Show that for values of $x$ less than unity

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \quad ....\right).$$

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### 6.3 Taylor expansion

The Taylor expansion is a series expansion of any well-behaved function such as usually encountered in physics. **Taylor's series** gives the value of a smooth function $f(x)$ in terms of the value of the function at $x = a$ and the values of the first and higher differentials of the function evaluated at $x = a$.

$$f(x) = f(a) + (x-a)\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)_a + \frac{(x-a)^2}{2!}\left(\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right)_a + \quad ... \tag{6.12}$$

In this formula the subscript $a$ on the differentials indicates that after their determination as a function of $x$ they are evaluated at $x = a$. The closer $a$ is to $x$ the more rapidly, in general, does the series converge, and as $(x-a)$ tends to the infinitesimally small interval $\mathrm{d}x$ we recover the definition of the differential given in equation 2.3.

The proof of Taylor's expansion is reasonably straightforward but lengthy and will not be pursued here. It can be found in standard mathematics texts.

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**Example 6.3** The potential energy between two atoms separated by a distance $r$ in a solid is often taken to be of the form

$$U(r) = \epsilon\left((\frac{a}{r})^{12} - (\frac{a}{r})^6\right),$$

with $a \approx 3 \times 10^{-10}$m and $\epsilon \approx 3 \times 10^{-20}$J. Find the mean distance apart, $a_0$, of the atoms. This is the distance which makes the potential energy a minimum. Show that for separations close to $a_0$ the potential energy may be closely approximated by

$$U(r) = U(a_0) + \frac{(r-a_0)^2}{2!}\left(\frac{\mathrm{d}^2 U}{\mathrm{d}r^2}\right)_{a_0}.$$

*Solution* the potential energy curve has a minimum given by $\mathrm{d}U/\mathrm{d}r = 0$.

$$\frac{\mathrm{d}U}{\mathrm{d}r} = \epsilon\left(\frac{-12a^{12}}{r^{13}} + \frac{6a^6}{r^7}\right).$$

Putting this derivative equal to zero gives the minimum at $r$ equal to $a$ times the sixth root of 2, or $a_0 = 1.13a = 3.4 \times 10^{-10}$m.

Using equation 6.12 to expand the potential energy around $a_0$

$$U(r) = U(a_0) + (r - a_0)\left(\frac{\mathrm{d}U}{\mathrm{d}r}\right)_{a_0} + \frac{(r-a)^2}{2!}\left(\frac{\mathrm{d}^2U}{\mathrm{d}r^2}\right)_{a_0} + ...$$

But $\mathrm{d}U/\mathrm{d}r = 0$ at $r = a_0$ hence

$$U(r) = U(a_0) + \frac{(r-a)^2}{2!}\left(\frac{\mathrm{d}^2U}{\mathrm{d}r^2}\right)_{a_0} + ...$$

with higher terms negligible if $r$ is close to $a_0$.

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**Problem 6.5** Two functions $f(x)$ and $g(x)$ are both zero at $x = 0$, hence the ratio $f/g$ as $x$ tends to zero can not be determined directly. L'Hospital's rule says that

$$\text{Limit as } x \to 0 \;\; \frac{f(x)}{g(x)} = \text{Limit as } x \to 0 \;\; \frac{\mathrm{d}f/\mathrm{d}x}{\mathrm{d}g/\mathrm{d}x}.$$

Prove the rule using Taylor expansions of $f$ and $g$.

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