Appendix A: Mathematical Formulas

Exponentials and Logarithms

$$e^{\ln x} = x, \quad \ln(e) = 1, \quad \ln(e^a) = a, \quad \ln(x^a) = a \ln x.$$
$$e^a e^b = e^{a+b}, \quad \ln(xy) = \ln x + \ln y, \quad \ln(x/y) = \ln x - \ln y$$

Solid Bodies

Volume of sphere radius R: $\dfrac{4\pi R^3}{3}$.

Surface area of sphere radius R: $4\pi R^2$.

Volume of cone, base R, height h: $\dfrac{\pi h R^2}{3}$.

Vectors

A vector can be expressed in terms of 3 orthogonal, normalised basis vectors which are commonly denoted $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ (or $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$): $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$, or represented by the number triplet of its components: $\mathbf{a} = (a_x, a_y, a_z)$.

The modulus of $\mathbf{a}$ is $|\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} = \sqrt{\mathbf{a} \cdot \mathbf{a}}$

Scalar product: if θ is the angle between two vectors $\mathbf{a}$ and $\mathbf{b}$, the scalar product is

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z = |\mathbf{a}||\mathbf{b}| \cos\theta.$$

Vector product: the direction of the vector product is normal to both $\mathbf{a}$ and $\mathbf{b}$ and has magnitude $|\mathbf{a}||\mathbf{b}| \sin\theta$.

Differentiation and Integration

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1} \qquad \int x^n \mathrm{d}x = \frac{1}{(n+1)} x^{n+1} + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^{ax}) = ae^{ax} \qquad \int e^{ax} \mathrm{d}x = \frac{1}{a} e^{ax}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln\, x) = \frac{1}{x} \qquad \int \frac{1}{x} \mathrm{d}x = \ln \mid x \mid + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(1/x) = -\frac{1}{x^2} \qquad \int \frac{1}{x^2} \mathrm{d}x = -\frac{1}{x} + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x \qquad \int \cos x \mathrm{d}x = \sin x + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \cos x = -\sin x \qquad \int \sin x \mathrm{d}x = -\cos x + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x \qquad \int \sec^2 x\mathrm{d}x = \tan x + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(a^x) = a^x \ln\, a \qquad \int a^x \mathrm{d}x = \frac{1}{\ln\, a}a^x + c$$

In the above a and c are constants with c the constant of integration.

- Product rule for differentiation

$$\frac{\mathrm{d}}{\mathrm{d}x}(f_1 f_2) = f_1 \frac{\mathrm{d}f_2}{\mathrm{d}x} + f_2 \frac{\mathrm{d}f_1}{\mathrm{d}x}$$

- Quotient rule for differentiation

$$\frac{\mathrm{d}}{\mathrm{d}x}(f_1/f_2) = \frac{f_2(\mathrm{d}f_1/\mathrm{d}x) - f_1(\mathrm{d}f_2/\mathrm{d}x)}{f_2^2}$$

- Integration by parts

$$\int f(x)\left(\frac{\mathrm{d}g(x)}{\mathrm{d}x}\right)\mathrm{d}x = f(x)g(x) - \int g(x)\left(\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right)\mathrm{d}x$$

Complex numbers

In terms of its modulus r and angle θ:

$$z = x + jy = re^{+j\theta} = r(\cos\theta + j\sin\theta).$$

Under complex conjugation $j \to -j$:

$$z^* = x - jy = r\ e^{-j\theta} = r(\cos\theta - j\sin\theta).$$

The modulus squared of z is the real number $|z|^2 = z^* z = x^2 + y^2 = r^2$.

Trigonometric functions

$$\sin^2 x + \cos^2 x = 1$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

The following may be derived easily from the above:

$$\sin x \cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y))$$
$$\cos x \cos y = \frac{1}{2}(\cos(x+y) + \cos(x-y)$$

$$\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y)$$

$$\sin x \pm \sin y = 2\sin\frac{1}{2}(x \pm y)\cos\frac{1}{2}(x \mp y)$$

$$\cos x + \cos y = 2\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$$

$$\cos x - \cos y = -2\sin\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y)$$

$$\sin 2x = 2\sin x\cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$

$$\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$

Series

- Geometric series:

$$1 + x + x^2 + \ldots + x^{n-1} = \frac{1-x^n}{1-x} \xrightarrow{n\to\infty} (1-x)^{-1} \qquad -1 < x < 1$$

- Binomial expansion:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

If n is a positive integer the series terminates after $n+1$ terms, otherwise it is an infinite series.

- Taylor Series about the point $x = x_0$:

$$f(x) = f(x_0) + (x-x_0)\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)_{x_0} + \frac{1}{2!}(x-x_0)^2\left(\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right)_{x_0} + ...$$

The following series are frequently encountered:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ...$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ....$$

$$\ln(1 \pm x) = \pm x - \frac{x^2}{2} \pm \frac{x^3}{3} - ....$$

$$(1 \mp x)^{-1} = 1 \pm x + x^2 \pm x^3 + ...\text{for} - 1 < x < 1.$$

L'Hospital's rule follows from Taylor's series: if $f(0) = g(0) = 0$,

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{\mathrm{d}f/\mathrm{d}x}{\mathrm{d}g/\mathrm{d}x}.$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Differential Equations

Important differential equations and their solutions:

$$\frac{\mathrm{d}y}{\mathrm{d}t} + \lambda y = 0 \qquad y = Ae^{-\lambda t}$$

$$a\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + cy = 0 \qquad y = B\cos\omega t + C\sin\omega t = A\cos(\omega t + \phi)$$

with $\omega = \sqrt{c/a}$

$$a\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0 \qquad y = Ae^{-\gamma t}\cos(\omega t + \phi)$$

where $a,\ b,\ c,\ A,\ B,\ C$ and ϕ are constants. In the last equation $\gamma = b/2a$ and $\omega = \sqrt{c/a - b^2/4a^2}$ and the solution given is correct for a, b and c all positive.

Partial Differentiation

Given a function f of more than one variable, the infinitesimal change in f when the variables are changed infinitesimally is

$$\mathrm{d}f(x,y) = \left(\frac{\partial f}{\partial x}\right)_y \mathrm{d}x + \left(\frac{\partial f}{\partial y}\right)_x \mathrm{d}y$$

Coordinate Systems

- **Cartesian coordinates:**$(x,\ y,\ z)$

The position vector

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

where x, y, and z are the cartesian coordinates of the point.

Volume element $\mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z$

- **Spherical polar coordinates:** (r,θ,ϕ)

$r = \sqrt{x^2 + y^2 + z^2}$; $\theta = \arctan(\sqrt{x^2 + y^2}/z)$; $\phi = \arctan(y/x)$;

$x = r\sin\theta\cos\phi$; $y = r\sin\theta\sin\phi$; $z = r\cos\theta$

Volume element: $dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$;

Area element on the surface of a sphere: $\mathrm{d}A = r^2 \sin\theta \, d\theta \, d\phi$

• **Cylindrical polar coordinates:** (r,ϕ,z)

Volume element: $\mathrm{d}V = r \ \mathrm{d}r, \ \mathrm{d}phi \ \mathrm{d}z$