1 Functions

The development of the mathematical language used in physics begins with the concept of the **function**. A quantity which changes value when the value of a second quantity, called a **variable**, changes is called a **function** of that variable. The speed V of a car depends on the time t at which it is measured: V is a function of t. The function is written $f(t)$.

$$V = f(t), \tag{1.1}$$

If we imagine instantaneous measurements of the speed to be made at vanishingly small intervals of time and plot how V varies with t, the graph constitutes a smooth curve. The curve shown on Figure 1.1 shows a car which speeds up then slows down less rapidly.

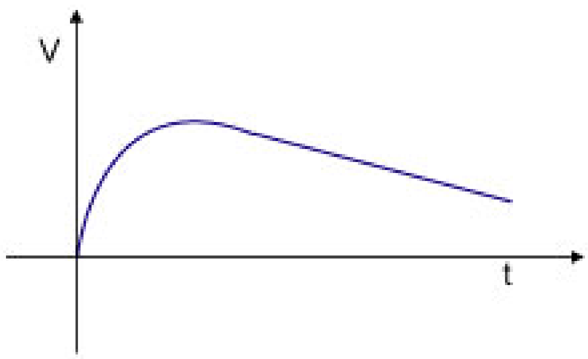

Figure 1.1 The speed of a car as a function of time.

Functions $f(x)$, of variables x, met in physics usually have a smooth dependence on x, and often in simple situations $f(x)$ can be written down algebraically in relatively simple terms. The simplest functions can be expressed in powers of x, each power multiplied by a **coefficient**. For example, a **linear** function, which corresponds to a straight-line dependence on the variable, can be written

$$y = f(x) = a + bx, \tag{1.2}$$

where the coefficient a gives the value of y at $x = 0$, and the coefficient b gives the increase in y for unit increase in x.

Problem 1.1. If the intercept on the y-axis is 4 and the increase in y when x increases by 4 is 20 what is the function?

The function

$$f(x) = a + bx + cx^2, \tag{1.3}$$

is a **quadratic** function and the sign and size of the coefficient c indicate whether the curve of $f(x)$ versus x bends upwards or downwards, and how rapidly, respectively.

Problem 1.2. Does the function $7-2x+3x^2$ bend upwards or downwards at large values of x?

Problem 1.3. A quadratic equation is obtained by putting equation 1.3 equal to zero. A certain quadratic equation may be solved to give the two values of x which satisfy it, called its **roots**. If α and β are the roots of the equation $x^2 - 6x + 4 = 0$, find the value of $\alpha^4 + \beta^4$.

More complicated functions may be be represented by adding terms involving higher powers of x, and the expression

$$f(x) = \sum_{n=0}^{n=N} a_n x^n, \tag{1.4}$$

is shorthand for

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ... + a_N x^N. \tag{1.5}$$

The summation symbol $\sum_{n=0}^{n=N}$ means add all terms like $a_n x^n$ for values of n from $n = 0$ to $n = N$ and a set of terms like equation 1.5 is called a **series**.

- **Functions of more than a single variable**

We have so far restricted attention to functions of a single variable only. However, a function can depend on several variables and physical quantities often depend on more than one. If a function f depends on x and y

$$f = f(x, y). \tag{1.6}$$

Now, if we wish to display graphically the variation of f with both x and y, it is necessary to have two horizontal axes along which x and y are displayed and the values of f fall on a surface. For example, the function $f = a - bx + cy$, with b and c positive constants, represents a plane with constant slope in any plane perpendicular to the y-axis, but as y increases from zero the value of f at $x = 0$ increases linearly. The function represents a tilted plane as shown in Figure 1.2.

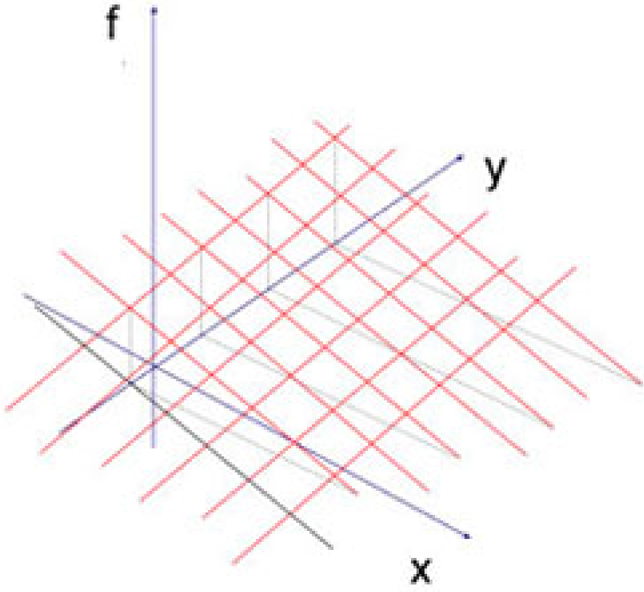

Figure 1.2 A function of two variables is a surface.

1.1 Scalars and Vectors

• A **scalar** is a physical quantity which can be specified by a single number alone. The mass of an object is a scalar. Several other examples immediately come to mind: the electrical charge on an object; the separation between two points; the speed of an object; the rate of consumption of electrical power; an interval of time, and so on. A scalar is defined by giving one number only: it has a size, or magnitude, and the number may have a sign. Some of the examples given above can be positive or negative, as, for example, the electric charge on an object.

• A **vector** is a physical quantity which requires both a magnitude and a direction for it to be defined completely. Examples of vectors are the velocity of an object, which needs speed and direction to tell which way it is going; force, which needs a number to tell how strong it is and a direction to tell the direction in which it is being applied, and so on. A quantity which is a vector will be indicated in this review by being in bold type.

• **Scalar and vector variables**.

One of the variables on which a function depends may be a vector, or a function may depend on a single vector alone. Indeed, a function may depend on a vector and on a scalar or depend on more than one variable of each type. There can be functions which are themselves vectors and be functions of scalars or can be scalars and be functions of vectors. The course is rarely concerned with such complications but it is useful to be aware of the versatility of the concept of function.

1.1.1 Addition and subtraction of vectors.

The arithmetical manipulation (addition, subtraction, multiplication and division) of numbers corresponding to scalars is straightforward. The manipulation of vectors is more complicated. The vector $(\mathbf{a}+\mathbf{b})$ is obtained as the diagonal of the parallelogram which has sides of lengths equal to the sizes a of vector $\mathbf{a}$ and b of vector $\mathbf{b}$, and directions corresponding to those of $\mathbf{a}$ and $\mathbf{b}$. This is shown in Figure 1.3.

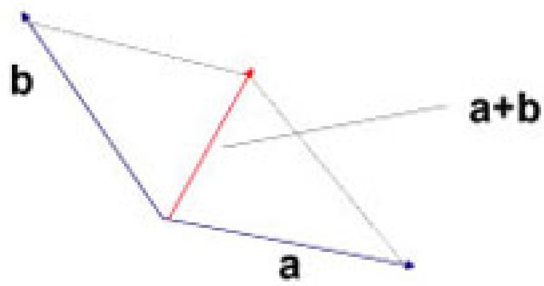

Figure 1.3 The addition of two vectors.

The sum is in the plane containing the two to be added and in the direction of the diagonal. Its magnitude often denoted by $|(\mathbf{a}+\mathbf{b})|$, is equal to the length of the diagonal.

The subtraction of a vector $\mathbf{b}$ from a vector $\mathbf{a}$, denoted $(\mathbf{a}-\mathbf{b})$, is obtained by adding to $\mathbf{a}$ the vector $-\mathbf{b}$, which is simply $\mathbf{b}$ reversed.

1.1.2 Components of vectors

A vector can be resolved into two components along chosen directions, or indeed into any number of components along convenient directions. Resolving a vector into two components is often useful in the treatment of problems, and the vectors $\mathbf{a}$ and $\mathbf{b}$ shown on Figure 1.3 are the components of the vector $(\mathbf{a}+\mathbf{b})$ in the directions of vectors $\mathbf{a}$ and $\mathbf{b}$.

1.1.3 Multiplication of vectors

There are two ways of multiplying vectors: one gives what is called the **scalar product**, the other the **vector product**. The scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is written $\mathbf{a}.\mathbf{b}$. As expected from the name, the scalar product is a scalar, and has magnitude equal to the product of the size of each vector and the cosine of the angle θ between them. (Trigonometric functions such as sine and cosine are considered in Section 5. They are often abbreviated as sin and cos, and always abbreviated in that way when they are included in formulae).

$$\mathbf{a}.\mathbf{b} = ab\ \cos\theta. \tag{1.7}$$

Problem 1.4 What is the scalar product of two vectors one of magnitude 4 cm, the other 5 cm? The angle between the vectors is 70°.

The vector product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is in turn a vector and is written $\mathbf{a} \times \mathbf{b}$. Its direction is perpendicular to the directions of both $\mathbf{a}$ and $\mathbf{b}$ and points the way given by the right hand rule. This rule, paraphrased, says that to determine the direction of $\mathbf{a} \times \mathbf{b}$ let your right foot point in the direction of $\mathbf{a}$ and your left foot in the direction of $\mathbf{b}$. The direction of $\mathbf{a} \times \mathbf{b}$ is now up through your head. This is shown in Figure 1.4. The magnitude of the vector product is equal to the product of the size of each vector and the sine of the angle between them.

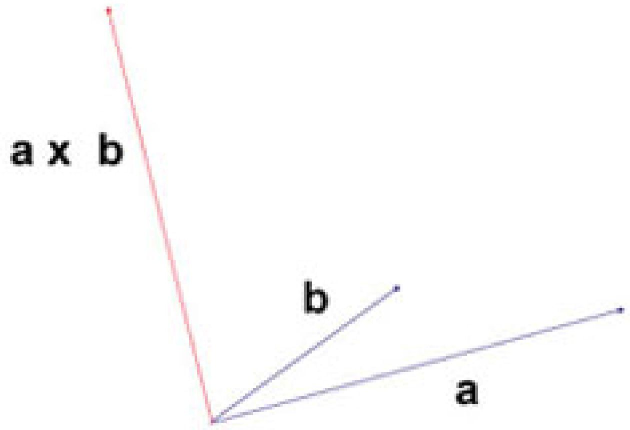

Figure 1.4 The vector product of two vectors.

Problem 1.5 What is the magnitude of the vector product of the two vectors in Problem 1.4?

There is no such thing as division of one vector by another. The expression $\mathbf{a}/\mathbf{b}$ has no meaning.