2 Differentiation

The differential calculus, discussed in this section, tells us what happens to a function as a variable on which the function depends is changed by smaller and smaller amounts, and finally by an infintesimally small amount.

When a variable x changes from x_1 to x_2 by $(x_2 - x_1)$ the function $f(x)$ changes from $f(x_1)$ to $f(x_2)$. If the change in x is smaller, and it changes by the very small amount δx a smoothly-varying function will change less. Suppose the interval δx becomes smaller and smaller. As it approaches zero the change in $f(x)$, which we will write $\delta f(x)$, also approaches zero. However, the ratio $\delta f(x)/\delta x$ can be non-zero. The **differential** of the function $f(x)$ at any point x is the limit of the above ratio as the interval δx becomes vanishingly small, when it is given the symbol $\mathrm{d}x$, and the vanishingly small increase in $f(x)$ is given the symbol $\mathrm{d}f(x)$.

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \text{Limit as } \delta x \text{ tends to zero of } \frac{\delta f(x)}{\delta x}. \tag{2.1}$$

The differential of a function at a point is the tangent to the curve at that point and gives the rate of change of the function with respect to the variable on which it depends. For example, Figure 2.1 shows a plot of the distance x travelled by a jogger as a function of time when the jogger accelerates to a constant speed. The plot of distance against time becomes uneven, although its trend is always increasing unless the person exercising decides to stop and run back the same way. The average speed over the time t_1 to t_2 is the total distance divided by the total time, $(x_2 - x_1)/(t_2 - t_1)$, but the speed at any instant of time varies. The instantaneous speed at time t_1 is obtained by taking the limit as δt becomes infinitesimally small of the ratio $\delta x/\delta t$, i.e. the differential of the function giving the dependence of x on time. The limit of this ratio is the slope or gradient of the curve at the time t_1. The differential calculus thus enables a more detailed description to be given of the jogger's motion than would be obtained by simply discussing average speeds over measurable time intervals.

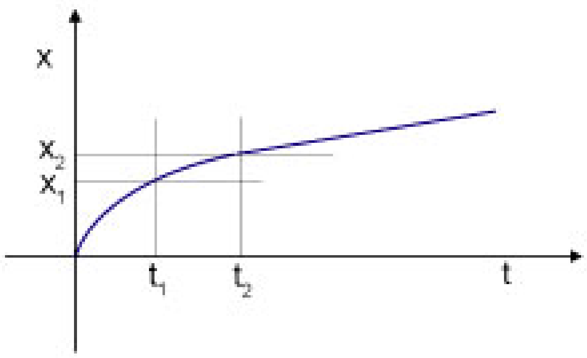

Figure 2.1 A plot of distance against time for a person running.

- **Maxima and Minima**

If the runner starts at time zero as in Figure 2.2, speeds up but then slows down to a stop at the point A, where the direction is reversed, the curve of distance against time shows a maximum at a time t_A. The rate of change of distance with time at A is the gradient of the curve at A, and there the gradient is zero. Setting the differential of a function equal to zero thus gives a method of finding the **maxima** of the function (if it has any). However, the gradient is also zero at **minima** of the function, at times such as t_B where the curve bends upwards after the runner has stopped at point B on the way back to the starting point and reversed direction once more. Hence, setting the differential to zero determines maxima and minima at the same time.

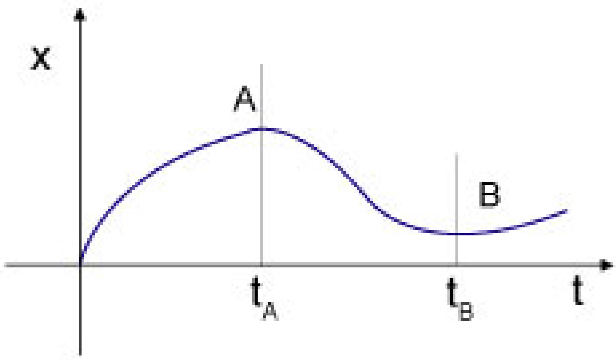

Figure 2.2 A plot of distance against time for a person running who does not always go the same way.

Problem 2.1 The **second differential** of a function $f(x)$, is simply the differential of the first differential, and is written $\mathrm{d}^2 f/\mathrm{d}x^2$.

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right). \tag{2.2}$$

Sketch the first and second differentials of the function in Figure 2.2 and show that the sign of the second differential determines whether a point is a maximum or a minimum.

Please note that no solution is provided for this problem.

2.1 Simple functions

We now show how to calculate the differentials of simple functions and give rules which help to do more complicated examples. The change $\mathrm{d}f(x)$ in $f(x)$ as x goes from x to $x + \mathrm{d}x$ is

$$\mathrm{d}f = f(x + dx) - f(x),$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{f(x + dx) - f(x)}{\mathrm{d}x}. \tag{2.3}$$

This is a general definition of the differentials of functions and can be used in their determination. When working out the numerator on the right-hand side of equation 2.3, only terms in $\mathrm{d}x$ need be retained; terms involving higher powers of $\mathrm{d}x$ can be neglected because if we kept them division by $\mathrm{d}x$ to give $\mathrm{d}f(x)/\mathrm{d}x$ would leave terms involving $\mathrm{d}x$ or higher powers of $\mathrm{d}x$. Since $\mathrm{d}x$ is vanishingly small these terms do not contribute to the differential.

Problem 2.2 The simplest example of differentiation is when the function is linear. (The differential of a constant function is of course zero). Show that the differential of the function given by equation (1.2) is $\mathrm{d}f/\mathrm{d}x = b$.

Problem 2.3 Consider now the quadratic function (1.3). Show that its differential is $\mathrm{d}f/\mathrm{d}x = b + 2cx$.

The method employed in the above problems (use of equation 2.3) can be used to show that

$$\frac{\mathrm{d}(ax^n)}{\mathrm{d}x} = anx^{(n-1)} \tag{2.4}$$

where a is a constant and n is any integer or number which can be expressed as the quotient of integers.

2.2 The exponential function

It can be shown that there is a number, e, which when raised to the power x produces a function, called the **exponential function**, e^x, whose differential is the same exponential function. This number is called the **base of natural logarithms**. Hence

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right) = e^x. \tag{2.5}$$

The numerical, value of e is 2.71828 to the nearest 5 decimal places. The exponential function may be written as a series in ascending powers of x.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ..., \tag{2.6}$$

with the series extending to infinity. The symbol $N!$ is shorthand for

$$N(N-1)(N-2)(N-3)....1. \tag{2.7}$$

and is called **factorial** N. (Note: The factorial of 0 is always equal to 1 ie $0! = 1$). Using equation 2.4 it may easily be seen that equation 2.6 satisfies equation 2.5.

- **The Differential of ln$x$**

The function ln x is the logarithm to the base e, or the **natural logarithm**, of x. If ln$x = y$, then $x = e^y$.

The special property of the exponential function leads to the result that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}. \tag{2.8}$$

2.3 Complicated functions

The technique illustrated above of using equation 2.3 can in principle be used to determine the differential of any function. Here we quote rules, obtained using equation 2.3, about finding the differentials of different kinds of complicated functions.

- **Powers of functions**

$$\frac{\mathrm{d}(f(x))^n}{\mathrm{d}x} = n(f(x))^{n-1} \times \frac{\mathrm{d}f(x)}{\mathrm{d}x}. \tag{2.9}$$

Example 2.1. Determine the first differential of the function $(3x^2 + 2x + 4)^{5/4}$.

Solution. In the notation used in equation 2.9, $f(x) = (3x^2 + 2x + 4)$ and $\mathrm{d}f(x)/\mathrm{d}x = 6x + 2$. Hence

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{5}{4}(3x^2 + 2x + 4)^{\frac{1}{4}} \times (6x + 2).$$

- **Products of functions**

If f_1 and f_2 are both functions of x,

$$\frac{\mathrm{d}(f_1.f_2)}{\mathrm{d}x} = f_1\frac{\mathrm{d}f_2}{\mathrm{d}x} + f_2\frac{\mathrm{d}f_1}{\mathrm{d}x}, \tag{2.10}$$

This category formally shows how to deal with constants which multiply a function. The constant can be ignored and reinstated to multiply the differential when this has been evaluated.

Example 2.2. Determine the first derivative of the function $f = x^{1/2}\ (1-x)^{-1/2}$.

Solution. In the notation used in equation 2.10, $f_1 = x^{1/2},\ f_2 = (1-x)^{-1/2}$.

$$\frac{\mathrm{d}f_1}{\mathrm{d}x} = \frac{1}{2}x^{-1/2}, \quad \frac{\mathrm{d}f_2}{\mathrm{d}x} = \frac{1}{2}(1-x)^{-3/2}.$$

Equation 2.10 now gives

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{1}{2}x^{-1/2}(1-x)^{-1/2} + \frac{1}{2}x^{1/2}(1-x)^{-3/2}.$$

Multiplying top and bottom by $(1-x)^{3/2}x^{1/2}$ this reduces to

$$\frac{1}{2(1-x)\sqrt{x(1-x)}}.$$

Problem 2.4. Determine the first differential of the function $(x^2 + \ln x)^2$.

- **Quotients of functions**

$$\frac{\mathrm{d}(f_1/f_2)}{\mathrm{d}x} = \frac{f_2(\mathrm{d}f_1/\mathrm{d}x) - f_1(\mathrm{d}f_2/\mathrm{d}x)}{f_2^2}. \tag{2.11}$$

Example 2.3. Determine the first derivative of the function
$f = x^{1/2}\ (1-x)^{-1/2} = \sqrt{\left(\frac{x}{(1-x)}\right)}$.

Solution. In the notation used in equation 2.11, $f_1 = x^{1/2}$ and $f_2 = (1-x)^{1/2}$. Equation 2.11 now gives

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{(1-x)^{1/2}\frac{1}{2}x^{-1/2} + x^{1/2}\frac{1}{2}(1-x)^{-1/2}}{(1-x)}.$$

Multiplying top and bottom by $x^{1/2}(1-x)^{1/2}$ this reduces to

$$\frac{1}{2}\frac{1}{(1-x)\sqrt{x(1-x)}},$$

as in Example 2.2.

Problem 2.5. Determine the second differential of the function $(x^2 + \ln x)^2$.

Problem 2.6 Find the maximum and the minimum values of the function $2x/(3 + x^2)$.
