3 Integration

An integral of a function gives another function. An integral of a function of a single variable x is defined as the sum of products of the values of the function at x and very small increments δx in x, in the limit as δx tends to the infinitesimal $\mathrm{d}x$.

Let the function $f(x)$ be the curve shown in Figure 3.1. The value of the area under the curve from $x = 0$ to $x = x_1$ is the sum of the areas of consecutive narrow strips of width δx and heights equal to the values of f at the ends of the strips. The x-coordinate x_n of the end of the n th strip is $n\delta x$, and the value of the function at this point is $f(n\delta x)$. The area under the curve between $x = 0$ and $x = x_1$ is given to a good approximation, if δx is small, by

$$\text{Area} = \sum_{n=0}^{n=(x_1/\delta x)} f(n\delta x)\delta x. \tag{3.1}$$

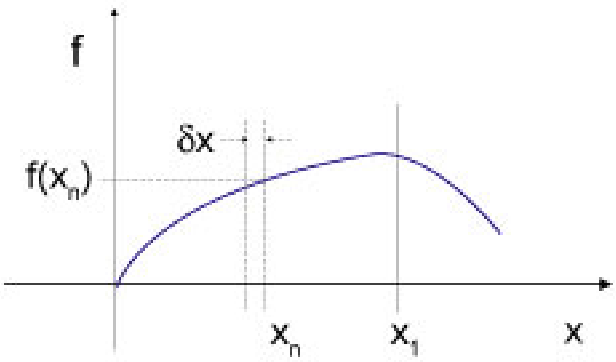

Figure 3.1

As the strip width δx tends to $\mathrm{d}x$, the sum of the areas of the strips becomes equal to the area under the curve and is the **integral** of f from $x = 0$ to $x = x_1$. The integral is then written as

$$\int_0^{x_1} f(x)\mathrm{d}x.$$

It is equal to the limit, as δx tends to $\mathrm{d}x$, of the right-hand side of equation 3.1 and may be written

$$\int_0^{x_1} f(x)\mathrm{d}x = \sum_{x=0}^{x=x_1} f(x)\mathrm{d}x. \tag{3.2}$$

Equation 3.2 defines a **definite integral**. The lower and upper limits on integral sign are the values of the variable x over which the integral is to be performed and the result depends on the function f and these limits. If the upper limit is regarded as a variable, the integral becomes a function of x, and we obtain an **indefinite integral**,

$$\int^x f(x)\mathrm{d}x.$$

When indefinite integrals are written down the limits are usually omitted.

- **Integration is the opposite of differentiation.**

Given a function $f(x)$ its **integral** is that function which, when differentiated, gives back $f(x)$.

Example 3.1 Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int^x f(x)\mathrm{d}x\right) = f(x).$$

Solution Let $J(x)$ denote the integral of $f(x)$.

$$J(x) = \int^x f(x)\mathrm{d}x = [\text{Limit as } \delta x \to \mathrm{d}x] \quad \sum^x f(x)\delta x.$$

From equation 2.3

$$\begin{aligned}\frac{\mathrm{d}J(x)}{\mathrm{d}x} &= [\text{Limit as } \delta x \to \mathrm{d}x] \quad \frac{J(x+\delta x) - J(x)}{\delta x}, \\ &= [\text{Limit as } \delta x \to \mathrm{d}x] \quad \frac{\sum^{x+\delta x} f(x)\delta x - \sum^x f(x)\delta x}{\delta x}, \\ &= [\text{Limit as } \delta x \to \mathrm{d}x] \quad \frac{f(x+\delta x)\delta x}{\delta x}, \\ &= \frac{f(x)\delta x + (\mathrm{d}f/\mathrm{d}x)\delta x^2}{\delta x} = f(x).\end{aligned}$$

3.1 Simple integrals

The simplest function whose integral can be determined is $f(x) = x$. From the formula giving the area of a triangle as half base times height, the integral of $f(x) = x$ over the range zero to x_1 is simply

$$\int_0^{x_1} x\mathrm{d}x = x_1^2/2. \tag{3.3}$$

Since integration is the reverse of differentiation, one way to find the indefinite integral of f is to find a function which when differentiated gives back f. It can easily be verified, for example, that

$$\int x^n \mathrm{d}x = \frac{1}{(n+1)}x^{n+1}. \tag{3.4}$$

It also follows from equation 2.8 that

$$\int \frac{1}{x}\,\mathrm{d}x = \ln(x), \tag{3.5}$$

where $\ln(x)$ is the natural logarithm of x. Any constant may be added to an indefinite integral because it vanishes on differentiation. Definite integrals do not involve arbitrary constants. If the value of the integral is known at a particular value of x, this determines the constant.

- **Some properties of integrals**

Since the area under the curve representing a function is the integral of the function, it is seen that

$$\int_{x_1}^{x_2} af(x)\mathrm{d}x = a\int_{x_1}^{x_2} f(x)\mathrm{d}x, \tag{3.6}$$

where a is a constant,

$$\int_{x_1}^{x_2}(f(x)+g(x))\mathrm{d}x = \int_{x_1}^{x_2} f(x)\mathrm{d}x + \int_{x_1}^{x_2} g(x)\mathrm{d}x, \tag{3.7}$$

and

$$\int_{x_1}^{x_2} f(x)\mathrm{d}x = \int_{x_1}^{x'} f(x)\mathrm{d}x + \int_{x'}^{x_2} f(x)\mathrm{d}x, \tag{3.8}$$

with x' between x_1 and x_2. It should also be noted that

$$\int_{x_1}^{x_2} f(x)\mathrm{d}x = -\int_{x_2}^{x_1} f(x)\mathrm{d}x. \tag{3.9}$$

Example 3.2 Show that the integral of the function $(4x^3 + 7x^2)$ is $(x^4 + 7x^3/3)$. The arbitrary constant has been put equal to zero.

Solution

$$\int (4x^3 + 3x^2)\mathrm{d}x = 4\int x^4\mathrm{d}x + 7\int x^2\mathrm{d}x,$$

from equations 3.6 and 3.7. Hence, from equation 3.4,

$$\int (4x^3 + 7x^2) = (x^4 + 7x^3/3).$$

Problem 3.1 Integrate $3x^2 - 3x + 8 - 1/x$. Show that adding the definite integral from 1 to 4 to the definite integral from 4 to 8 gives the same answer as the definite integral from 1 to 8. Show that the definite integral from 1 to 8 equals the definite integral from 1 to 10 less the definite integral from 8 to 10.

• **Integration by substitution**

The integral of a function $f(x)$ over a variable x can be changed into an integral over a new variable y by making x a convenient function of $y : x = g(y)$. Now

$$\int_{x_1}^{x_2} f(x)\mathrm{d}x = \int_{y(x_1)}^{y(x_2)} f(x)\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)\mathrm{d}y. \tag{3.10}$$

This topic is revisited with examples in the next section after differentiation and integration of trigonometric functions have been discussed. It is placed here for completeness.

• **Integration by parts**

If an integrand is the product of two functions, one of which is $f(x)$ and the other is the differential of a function $g(x)$

$$\int f(x)\left(\frac{\mathrm{d}g(x)}{\mathrm{d}x}\right)\mathrm{d}x = f(x)g(x) - \int g(x)\left(\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right)\mathrm{d}x. \tag{3.11}$$

Example 3.3 Show that

$$\int xe^{2x}\mathrm{d}x = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}$$

and verify the answer by differentiation.

Solution

In the notation of equation 3.11 put $f(x) = x$ and $\mathrm{d}g(x)/\mathrm{d}x = e^{2x}$, when $g(x) = \frac{1}{2}e^{2x}$ and $\mathrm{d}f(x)/\mathrm{d}x = 1$. Equation 3.11 now gives

$$\begin{aligned}\int xe^{2x}\mathrm{d}x &= \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\mathrm{d}x \\ &= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}.\end{aligned}$$

The arbitrary constant in the above has been ignored.

The differential of the first term in the answer is given by equation 2.10 as $\frac{1}{2}e^{2x} + xe^{2x}$. The differential of the second term is $-\frac{1}{2}e^{2x}$ and adding recovers xe^{2x}, the expression which was integrated.

Problem 3.2 Show that

$$\int x^2e^{2x}\mathrm{d}x = \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + \text{constant}.$$

• **Integrals occur throughout physics.** Many physical problems involve evaluating the sum of an infinite number of vanishingly small contributions, i.e. they involve an integral. Lists of useful definite and indefinite integrals are given in the formulae section.

3.2 Double and triple integrals

The discussion above was limited to the integration of functions of a single variable. The integral of a function of two variables is a double integral, and a triple integral corresponds to the integration of a function of three variables. The double integral of a function $f(x, y)$ is

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y) \mathrm{d}y \mathrm{d}x.$$

The integral is evaluated by performing one of the integrations first, keeping the other constant and then performing the second integration. This is in principle straightforward if the limits on one of the variables do not involve the other. If however that is not the case the integration is more difficult.

Example 3.4 Show that

$$\int_{x_1=0}^{x_2=a} \int_{y_1=0}^{y_2=a} (x^2 + y^2) \mathrm{d}y \mathrm{d}x = \frac{2}{3} a^4.$$

Solution Performing the x-integration first gives

$$\begin{aligned}\int_0^a \int_0^a (x^2 + y^2) \mathrm{d}x \mathrm{d}y &= \int_0^a \left(\frac{1}{3} x^3 + xy^2 \right)_0^a \mathrm{d}y \\ &= \int_0^a \left(\frac{1}{3} a^3 + ay^2 \right) \mathrm{d}y \\ &= \left(\frac{1}{3} a^3 y + \frac{1}{3} ay^3 \right)_0^a\end{aligned}$$

which is $\frac{2}{3} a^4$.

Example 3.5 Integrate the function $(16 - x^2 - y^2)$ for values of x between 0 and 2 and values of y between 0 and x.

Solution Since the upper limit on y depends on x the y integral must be done first keeping x constant.

$$\begin{aligned}\int_0^2 \int_{y=0}^{y=x} (16 - x^2 - y^2) \mathrm{d}y \mathrm{d}x &= \int_0^2 \left(16y - x^2 y - \frac{1}{3} y^3 \right)_0^x drx \\ &= \int_0^2 \left((16x - x^3 - \frac{1}{3} x^3 \right) \mathrm{d}x \\ &= \int_0^2 \left(16x - \frac{4}{3} x^3 \right) \mathrm{d}x \\ &= \left(8x^2 - \frac{1}{3} x^4 \right)_0^2,\end{aligned}$$

giving the value 80/3.

Problem 3.3 Show that

$$\int_{x=0}^{x}\int_{y=0}^{y=x+1}(4xy+3y^2)\mathrm{d}y\mathrm{d}x = \frac{3}{4}x^4+\frac{7}{3}x^3+\frac{5}{2}x^2+x.$$

Triple integrals are dealt with in a similar way to the treatment of double integrals discussed above. Examples of double and triple integrals when the variables are coordinates of a point in different coordinate systems are given in Section 9.