

4 Complex numbers

Real numbers relate to our normal world of experience. The square root of 4 is plus or minus 2 and the square root of 17 is plus or minus 4.1231 to four places of decimals. What is the square root of -17? The square roots of negative numbers can be interpreted by defining the square root of -1 to be the purely **imaginary** number j , such that $j^2 = -1$. This device extends the mathematical tools used in physics and enables many problems to be treated in a convenient fashion.

A **complex number** z normally has both a real part a and an imaginary part jb , and is simply an ordered pair of two real numbers a and b . It is written

$$z = (a + jb). \quad (4.1)$$

Figure 4.1 is an **Argand diagram** in which the real part of a complex number is plotted along the x axis and the imaginary part along the y axis. Any point off the x -axis thus corresponds to an imaginary number and the $x - y$ plane may be regarded as an imaginary plane. An imaginary number $z = (a + jb)$ is located in this plane at the point P with $x = a; y = b$. We may write the number z in terms of the angle θ the line PO makes with the positive x -axis and the distance r from the origin O at $x = 0; y = 0$ to the point P.

$$z = r(\cos \theta + j \sin \theta). \quad (4.2)$$

In this notation, r is called the **modulus** or **magnitude** of z , and θ is called its **argument**.

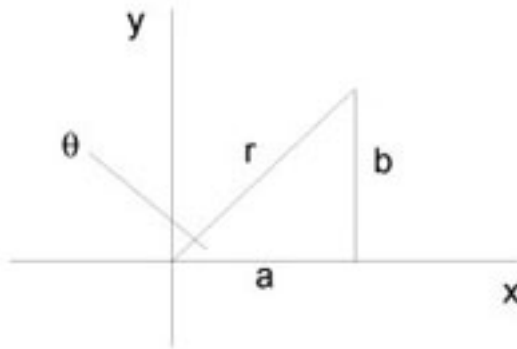


Figure 4.1

From equations (4.1) and (4.2), $a = r \cos \theta$ and $b = r \sin \theta$. Using the identity $\sin^2 \theta + \cos^2 \theta = 1$ given in Section 5,

$$r = \sqrt{a^2 + b^2}, \quad (4.3)$$

and

$$\tan \theta = b/a. \quad (4.4)$$

Example 4.1 Determine the modulus of the imaginary number $(3 - 4j)$

Solution Equation 4.3 gives

$$r = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5.$$

The **complex conjugate** of an imaginary number is obtained by changing the signs of all imaginary terms. Often the best way of determining the modulus of a complicated complex number is to multiply the number by its complex conjugate and take the square root of the answer. The complex conjugate of the complex number z is given the symbol z^* .

$$r = \sqrt{zz^*}. \tag{4.5}$$

• Addition and Subtraction

Addition and subtraction of two complex numbers is performed by adding or subtracting their real parts and their imaginary parts.

$$(a_1 + jb_1) + (a_2 + jb_2) = (a_1 + a_2) + j(b_1 + b_2). \tag{4.6}$$

$$(a_1 + jb_1) - (a_2 + jb_2) = (a_1 - a_2) + j(b_1 - b_2). \tag{4.7}$$

• Multiplication

Multiplication is performed in the same way algebraic expressions are multiplied.

$$\begin{aligned} (a_1 + jb_1) \times (a_2 + jb_2) &= a_1a_2 + a_1jb_2 + jb_1a_2 + jb_1jb_2 \\ &= (a_1a_2 - b_1b_2) + j(a_1b_2 + a_2b_1). \end{aligned} \tag{4.8}$$

Problem 4.1 Show that the modulus of the product of the two imaginary numbers $(2+3j)$ and $(1 - 2j)$ is $\sqrt{65}$.

• Division

Division of two imaginary numbers is performed by first multiplying the denominator and numerator by the complex conjugate of the denominator thus making the denominator real. The complex conjugate of $(a + jb)$ is $(a - jb)$ with

$$(a + jb) \times (a - jb) = (a^2 + b^2)$$

from equation 4.8.

Example 4.2 Determine the complex number obtained by dividing $(3 + 5j)$ by $(2 - 3j)$.

Solution

$$\frac{(3 + 5j)}{(2 - 3j)} = \frac{(3 + 5j)(2 + 3j)}{(2 - 3j)(2 + 3j)} = \frac{(-9 + 19j)}{13} = (-9/13 + 19j/13).$$

Problem 4.2 Show that

$$\frac{(3 + 4j)(1 - 2j)}{(2 - 3j)(2 + j)} = (17 + 6j)/13.$$
