5 Trigonometric functions

Trigonometry is the mathematics of triangles. A right-angled triangle is one in which one angle is 90°, as shown in Figure 5.1. The third angle in the triangle is $\phi = (90° - \theta)$.

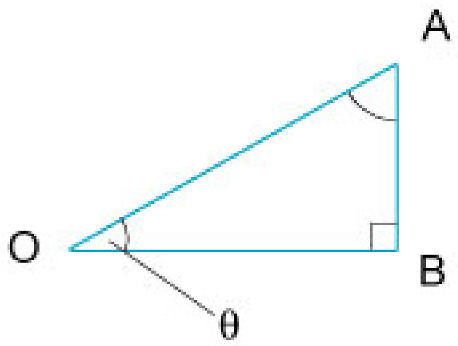

Figure 5.1

Six ratios can be constructed involving the three sides of a right-angled triangle and these depend only on the angle θ. The ratios are functions of the variable θ and are called **trigonometric functions**. We have already assumed familiarity with the basic trigonometric functions, sine, cosine and tangent but we list below all six for completeness.

$$\sin\ \theta = \frac{AB}{OA}, \tag{5.1}$$

$$\cos\ \theta = \frac{OB}{OA}, \tag{5.2}$$

$$\tan\ \theta = \frac{AB}{OB} = \frac{\sin\theta}{\cos\theta}, \tag{5.3}$$

$$\text{cosec}\ \theta = \frac{1}{\sin\theta}, \tag{5.4}$$

$$\sec\ \theta = \frac{1}{\cos\theta}, \tag{5.5}$$

$$\text{cotan}\ \theta = \frac{1}{\tan\theta}. \tag{5.6}$$

The length of the hypotenuse is always positive and the signs of the lengths of the sides encompassing the right angle depend in the normal way on which side of the point O the points A and B lie. With this convention the sines of angles between 90° and 180° are positive. For an angle $\pi - \theta$ greater than 90°, Figure 5.2, in which the length AB equals the length A′B′ and the length OB equals the length OB′, shows that $\sin(\pi - \theta)$ =A′B′/OA′=AB/OA $= \sin\theta$. The sines of angles between 180° and 360° are negative, and the signs of cosines are negative between 90° and 270°, becoming positive between 270° and 360°. As the angle θ is increased to 90°, $\sin\theta$ approaches unity.

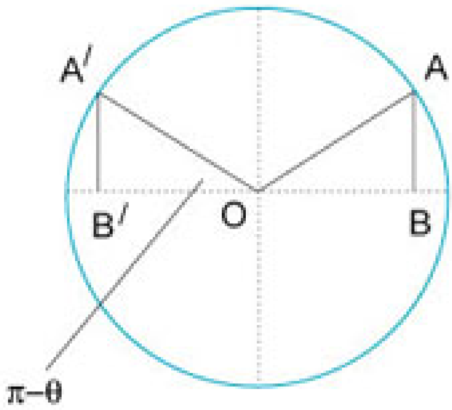

Figure 5.2

Example 5.1 Determine $\sin 210°$, $\cos 120°$ and $\tan 330°$.

Solution Using figures similar to 5.1 with sides 2,1 and $\sqrt{3}$, $\sin 210° = -\sin 30° = -1/2$, $\cos 120° = -1/2$ and $\tan 330° = -1/\sqrt{3}$.

5.1 Trigonometric relationships

Many relationships can be derived between the trigonometric functions using Euclidian geometry. For example, Pythagoras' theorem tells us that the square of the hypotenuse in a right-angled triangle equals the sum of the squares of the other two sides, and this immediately leads to the relation

$$\sin^2\theta + \cos^2\theta = 1, \tag{5.7}$$

for any angle θ.

The following equations relating the angles θ and ϕ may also be proved.

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi, \tag{5.8}$$

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi, \tag{5.9}$$

where θ and ϕ are any two angles, and $\mp$ means for $\cos(\theta + \phi)$ use the minus sign on the right-hand side, and for $\cos(\theta - \phi)$ use the plus sign.

Many useful relations between trigonometric functions can be obtained using equations 5.8 and 5.9 and simple algebraic manipulation.

Problem 5.1 Use equations 5.8 and 5.9 to determine $\sin 210°$, $\cos 120°$ and $\tan 330°$.

Example 5.2 Show that

$$\sin\theta = \frac{2\tan(\theta/2)}{1+\tan^2(\theta/2)}.$$

Solution

$$\sin\theta = \sin(\theta/2+\theta/2) = 2\sin(\theta/2)\cos(\theta/2) = \frac{2\sin(\theta/2)\cos(\theta/2)}{\sin^2(\theta/2)+\cos^2(\theta/2)},$$

from equations 5.8 and 5.7. Dividing top and bottom by $\cos^2(\theta/2)$ gives the answer.

Problem 5.2 Show that

$$\tan\theta = \frac{2\tan(\theta/2)}{1-\tan^2(\theta/2)}.$$

Problem 5.3 Using suitable constructions it is straightforward to show that if a triangle has sides a, b and c and the angles opposite the sides are α, β and γ,

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

Use this as the starting point and derive equation 5.8.

5.2 Differentials and Integrals

Use can be made of the above equations (5.8) and (5.9) to determine the differentials and hence the integrals of the trigonometric functions.

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(\sin\ \theta) = \frac{1}{\delta\theta}\left(\sin(\theta+\delta\theta) - \sin\ \theta\right),$$

in the limit as $\delta\theta$ tends to the vanishingly small $\mathrm{d}\theta$. Hence, from equation (5.8),

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(\sin\theta) = \frac{1}{\mathrm{d}\theta}\ (\sin\theta\cos\mathrm{d}\theta + \cos\ \theta\sin\mathrm{d}\theta - \sin\ \theta),$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(\sin\theta) = \cos\ \theta. \tag{5.10}$$

To obtain the last equation we have used the fact that as $\delta\theta$ becomes the infinitesimally small $\mathrm{d}\theta$, $\sin\ \mathrm{d}\theta$ becomes $\mathrm{d}\theta$ and $\cos\mathrm{d}\theta$ becomes unity, the cosine of a vanishingly small number.

In a similar way, using equation (5.8), it can be shown that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(\cos\theta) = -\sin\theta, \tag{5.11}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(\tan\theta) = \sec^2\theta, \tag{5.12}$$

where the last relation can be derived using equation (5.3).

Example 5.3 Show that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(\sin^3\theta\cos\theta) = \sin^2\theta(4\cos^2\theta - 1).$$

Solution

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}(\sin^3\theta\cos\theta) &= 3\sin^2\theta\cos^2\theta - \sin^4\theta = \sin^2\theta(3\cos^2\theta - \sin^2\theta)\\ &= \sin^2\theta(4\cos^2\theta - 1).\end{aligned}$$

The solution used equations 2.10, 5.10 and 5.11.

Problem 5.4 Show that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\sin\theta\cos\theta}{1+\sin\theta}\right) = \frac{\cos^2\theta - \sin^2\theta - \sin^3\theta}{(1+\sin\theta)^2}.$$

The integrals of the trigonometric functions can be obtained as the reverse of the differentials, giving

$$\int \sin\theta\ \mathrm{d}\theta = -\cos\theta, \tag{5.13}$$

$$\int \cos\theta\ \mathrm{d}\theta = \sin\theta, \tag{5.14}$$

and

$$\int \tan\theta\ \mathrm{d}\theta = -\ln(|\cos\theta|) = \ln(|\sec\theta|). \tag{5.15}$$

The last relation is not so obvious as the first two but can readily be verified by differentiation.

Several differentials and integrals of trigonometric expressions are given in the formulae section. One technique for performing integrals is the method of substitution of variables first mentioned in Section 3.1.

Example 5.4 Integrate $\int \sin^2\theta \cos\theta \mathrm{d}\theta$.

Solution

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\theta = \cos\theta,$$

and

$$\mathrm{d}(\sin\theta) = \cos\theta \mathrm{d}\theta.$$

Hence

$$\int \sin^2\theta \cos\theta \mathrm{d}\theta = \int \sin^2\theta \ \mathrm{d}(\sin\theta) = \frac{1}{3}\sin^3\theta,$$

ignoring the arbitrary constant.

Problem 5.5 Show that

$$\int \sin^2\theta \cos^3\theta \mathrm{d}\theta = \frac{\sin^3\theta\cos^2\theta}{5} + \frac{2}{15}\sin^3\theta.$$

- **Trigonometric functions as exponentials**

The exponential function $e^{j\theta}$ where θ is a real number is complex. The square of its modulus, and thus its modulus, is unity.

$$e^{j\theta} \times e^{-j\theta} = 1.$$

The complex number

$$z = \cos\theta + j\sin\theta,$$

also has unit modulus, and it can be shown that

$$e^{j\theta} = \cos\theta + j\sin\theta. \tag{5.16}$$

This connection between the exponential function with imaginary argument and sines and cosines is extremely useful. Since $\sin(-\theta) = \sin(0-\theta) = -\sin\theta$, from equation (5.8), and $\cos(-\theta) = \cos(0-\theta) = \cos\theta$ from equation 5.9,

$$e^{-j\theta} = \cos\theta - j\sin\theta. \tag{5.17}$$

Using the last two equations gives

$$\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta}), \tag{5.18}$$

and

$$\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta}) \tag{5.19}$$

Example 5.5 Show that equations 5.16 and 5.17 satisfy equation 5.8, thus establishing the validity of that equation.

Solution

$$\sin(\theta+\phi) = \frac{1}{2j}(e^{j(\theta+\phi)} - e^{-j(\theta+\phi)}).$$

$$\sin\theta\cos\phi = \frac{1}{4j}(e^{j\theta} - e^{-j\theta})(e^{j\phi} + e^{-j\phi})$$

$$= \frac{1}{4j}(e^{j(\theta+\phi)} + e^{j(\theta-\phi)} - e^{-j(\theta-\phi)} - e^{-j(\theta+\phi)}).$$

$$\cos\theta\sin\phi = \frac{1}{4j}(e^{j\theta} + e^{-j\theta})(e^{j\phi} - e^{-j\phi})$$

$$= \frac{1}{4j}(e^{j(\theta+\phi)} - e^{j(\theta-\phi)} + e^{-j(\theta-\phi)} - e^{-j(\theta+\phi)}).$$

Hence

$$\sin\theta\cos\phi + \cos\theta\sin\phi = \frac{1}{2j}(e^{j(\theta+\phi)} - e^{-j(\theta+\phi)})$$

$$= \sin(\theta+\phi).$$

Problem 5.6 Show that equations 5.16 and 5.17 satisfy equation 5.9.

Problem 5.7 Show that

$$\sin\theta + \sin\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right),$$

and that

$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right).$$

• Polar form of complex numbers

Using equation (5.16) we may now write the complex number $z = a + jb$ in its **polar** form. If a complex number $z = a + jb$ has a modulus r,

$$z = (a + jb) = re^{j\theta} \qquad (5.20),$$

where r is given by equation 4.3 and the argument θ is given by equation 4.4. It is often useful to express complex numbers in polar form, when multiplication becomes addition of the arguments of exponentials.

Problem 5.8 Express $z = (2-3j)(1+2j)/(4+3j)$ in polar form.
