7 Differential equations

Most of the physical laws extracted from experimental observations are expressed mathematically in terms of differential equations. A simple example is when an observable y is a function of a single variable x and the experimental observations indicate a relationship between y and its first and/or higher differentials with respect to x. Such a relationship is a **differential equation**, and the solution of the equation gives the function y and determines how it varies with x.

Suppose there is a fixed amount of radioactive isotope which decays to a neighbouring stable nucleus. The radioactive material is divided into batches of different mass and the number of decays from each batch measured over the same time interval. The data show that the numbers of decays are proportional to the mass of material used, whatever the time interval over which the measurements on the different samples were made, and also show that as the time interval is varied the numbers of decays are proportional to the size of that interval. These observations suggest that the number of decays $\mathrm{d}N$ in an infinitesimal time interval $\mathrm{d}t$ is proportional to the number of radioactive atoms present and to the interval $\mathrm{d}t$.

$$\mathrm{d}N = -\lambda N \mathrm{d}t,$$

or

$$\frac{\mathrm{d}N}{N} = -\lambda \mathrm{d}t, \tag{7.1}$$

where λ is the constant of proportionality. The minus sign is present because the decays $\mathrm{d}N$ represent a reduction in the number N of nuclei.

Integration of both sides of equation 7.1 gives

$$\ln N = -\lambda t + \text{constant},$$

and if the number of atoms present at time zero is N_0, the constant is $\ln N_0$ and

$$N = N_0 \exp(-\lambda t), \tag{7.2}$$

the familiar law of exponential decay. The constant λ is called the **decay constant** and is a characteristic of the particular decaying species.

Problem 7.1 Show that half of an original sample has decayed after a time equal to the **half life** given by $\tau_{1/2} = \ln 2/\lambda$.

Problem 7.2 A beam of I_0 gamma rays is incident upon a thin metal sheet of thickness d. The number of gamma rays lost from the beam in a thin element of thickness δx of the metal is proportional to the number $I(x)$ present at the position of the element and the thickness δx. Determine the number of gamma rays emerging from the metal in terms of I_0, d and the constant of proportionality γ.

7.1 First-order equations

Equation (7.1) is called a **first-order** differential equation because it involves only the first derivative with respect to time of the function N whose behaviour the equation describes. The solutions of first-order equations include one constant of integration which is determined by specifying the value of the function for a given value of the variable.

Equation 7.1 is a simple first-order equation, but more complicated examples often occur. A **linear** first-order equation in is one in which there are no terms involving powers of y and $\mathrm{d}y/\mathrm{d}x$ higher than the first. A general linear first-order equations has the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + ky = f(x), \tag{7.3}$$

where k is a constant. This is solved by multiplying both sides of equation 7.3 by the **integrating factor**

$$\beta(x) = e^{kx}. \tag{7.4}$$

The new left-hand side is

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x} + ky\right) \times e^{kx} = \frac{\mathrm{d}}{\mathrm{d}x}\left(ye^{kx}\right)$$

and equation 7.3 becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(ye^{kx}\right) = f(x)e^{kx}.$$

Integrating both side gives

$$ye^{kx} = \int f(x)e^{kx}\mathrm{d}x + C,$$

where C is a constant of integration, and

$$y = e^{-kx}\int f(x)e^{kx}\mathrm{d}x + Ce^{-kx}. \tag{7.5}$$

Problem 7.3 A non-linear first-order equation is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N^2.$$

Show that, if $N = N_0$ at $t = 0$,

$$N = N_0(1 + N_0\lambda t)^{-1}.$$

Example 7.1 A radioactive source with decay constant λ_1 decays to a daughter nucleus which in turn decays with a decay constant λ_2. If there are N_0 parent nuclei at time $t = 0$ and the number of daughter nuclei is zero at that time, the number $N_2(t)$ of daughter nuclei at time t is not equal to the number of decays of the parent nucleus because some of the daughters produced in the interval between times zero and t have decayed. Determine the number N_2 as a function of time.

Solution The differential equation which can be solved to give $N_2(t)$ can be established by noting that at time t, in the very small time interval δt, the increase in N_2 equals the number of parent decays minus the number of daughter decays.

$$\delta N_2 = \lambda_1 N_1 \delta t - \lambda_2 N_2 \delta t,$$

where N_1 is the number of parent nuclei at time t. In the limit as δt tends to the infinitesimal $\mathrm{d}t$ we obtain the differential equation

$$\frac{\mathrm{d}N_2}{\mathrm{d}t} = \lambda_1 N_1 - \lambda_2 N_2,$$

and from equation 7.2

$$\frac{\mathrm{d}N_2}{\mathrm{d}t} + \lambda_2 N_2 = \lambda_1 N_0 e^{-\lambda_1 t}. \tag{7.6}$$

This equation has the form of equation 7.3 and the technique of using an integrating factor described above can be used to obtain its solution. In the notation of equation 7.3, $k = \lambda_2$ and the integrating factor is $e^{\lambda_2 t}$. Multiplying both side of equation 7.5 by this factor gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(N_2 e^{\lambda_2 t}\right) = \lambda_1 N_0 e^{(\lambda_2 - \lambda_1)t}.$$

Integrating both sides

$$N_2 = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + C e^{-\lambda_2 t},$$

where C is a constant to be determined from the initial condition that at time zero $N_2 = 0$. This gives

$$C = -\frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}$$

and

$$N_2(t) = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t}).$$

Problem 7.4 A first-order equation which describes the time variation of the electric current $I(t)$ though a coil which has an inductance L and resistance R has the form

$$L\frac{\mathrm{d}I(t)}{\mathrm{d}t} + RI(t) = V(t).$$

If $V(t)$ is a constant voltage V_0 applied at time zero, show that

$$I(t) = \frac{V_0}{R}(1 - e^{-Rt/L}).$$

7.2 Second-order equations

More complicated differential equations for a function $y(x)$ of one variable x may involve double differentials, $\mathrm{d}^2y/\mathrm{d}x^2$, as well as the first differential $\mathrm{d}y/\mathrm{d}x$. The equation is then a **second-order** differential equation. The general form of a linear second-order differential equation is

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x), \tag{7.7}$$

where a, b and c are constants and $f(x)$ is a function of the variable x. We note that any function $g(x)$ which satisfies the equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0, \tag{7.8}$$

may be added to the solution $y = h(x)$ of equation 7.7 to give a function $h(x)+g(x)$ which is also a solution. The function $g(x)$ is called the **complementary function** and $h(x)$ is called the **particular integral**. Second-order equations involve two constants of integration which are determined by specifying values of y for two values of x. These constants are in the complementary function.

• Equation 7.7 simplifies if both $f(x)$ and the constant b are zero. Then

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + cy = 0. \tag{7.9}$$

If both a and c are greater than zero, the solution is

$$y = A\sin\omega x + B\cos\omega x \tag{7.10}$$

with $\omega = \sqrt{c/a}$, as can readily be verified by substitution. In the above, A and B are the two constants determined by the conditions imposed on the problem.

If a is greater than zero but c is less than zero, put $d = -c$ making d positive. Equation 7.9 becomes

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - dy = 0 \tag{7.11}$$

with solution

$$y = Ae^{\omega x} + Be^{-\omega x} \tag{7.12}$$

with $\omega = \sqrt{d/a}$.

Problem 7.5 Equation 7.10 can be rewritten in the form

$$y = \alpha\sin(\omega x + \beta).$$

Determine α and β.

• There are formal techniques for solving equation 7.7 when $f(x)$ is non-zero and for solving more complicated differential equations. Many solutions can be found using intelligent guesswork based on an understanding of the physics of the situation. Substitution of proposed solutions into equations will determine whether they are correct or not.

Problem 7.6 Show that, if a, b and c are positive and $f(x) = 0$, the solution to equation 7.7 is $y = \alpha \sin(\omega x + \beta)$ multiplied by the factor $e^{-\gamma x}$ with $\gamma = b/2a$ and $\omega = \sqrt{c/a - b^2/4a^2}$.

Example 7.2 The equation which determines the displacement y of a damped oscillator driven by a sinusoidal force is

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \gamma \frac{\mathrm{d}y}{\mathrm{d}t} + ky = b \sin \omega t, \tag{7.13}$$

where $\gamma,\ k$ and b are positive constants. Determine $y(t)$ in the steady state.

Solution In the steady state the complementary function, which includes the exponentially decaying factor corresponding to that in Problem 7.6, will have died away. The motion will be oscillatory with the same angular frequency as the force and an amplitude A. However, it may be out of phase. In the light of this, try as a solution

$$y = A \sin(\omega t + \theta)$$

with θ the angle by which the sinusoidal displacement y leads the force.

Inserting the first and second differentials of the assumed solution y into equation 7.13 gives

$$-A\omega^2 \sin + A\gamma\omega \cos(\omega t + \theta) + A \sin(\omega t + \theta) = b \sin \omega t. \tag{7.14}$$

For the assumed y to be the solution, equation 7.14 must hold at all times after the decay of the complementary function, and choosing two different times such that $(\omega t + \theta)$ is an integral multiple of $\pi/2$ and then an integral multiple of π, gives the following two simultaneous equations.

$$A\omega\gamma = b \sin(-\theta) = -b \sin \theta,$$

and

$$-A\omega^2 + Ak = b \sin(\pi/2 - \theta) = b \cos \theta.$$

Solving for A and θ using equation 5.7 gives

$$A = \frac{b}{\sqrt{(k - \omega^2)^2 + \omega^2 \gamma^2}}$$

and $\tan \theta = -\omega\gamma/(k - \omega^2)$. If the assumed solution had not been correct for suitable A and θ, the substitution into equation 7.13 would not have worked.

Problem 7.7 Several simple second-order equations can be solved by assuming a solution of the form of a series in ascending powers of x with coefficients to be determined and stopping the series at a suitable point. Show, in this way, that the solution of the equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \alpha y = x^2,$$

is

$$y = -\frac{2}{\alpha^2} + \frac{x^2}{\alpha}.$$