Solutions to Problems

Problem 7.1

When one half the sample has decayed, $N = N_0/2$ and equation 7.2 gives the time $t_{1/2}$ as the solution to $N_0/2 = N_0 e^{-\lambda t_{1/2}}$. Hence $e^{\lambda t_{1/2}} = 2$ and $t_{1/2} = \ln 2/\lambda$.

Problem 7.2

$$\mathrm{d}I = \lambda I \mathrm{d}x \;\; \text{hence} \;\; \frac{\mathrm{d}I}{I} = -\lambda \mathrm{d}x.$$

As for equation 7.1 the solution is $I = I_0 e^{-\lambda x}$ and thus when $x = d$ the number of gamma rays in the beam is $I(d) = I_0 e^{\lambda d}$.

Problem 7.3

Integrating both sides of $\mathrm{d}N/N^2 = -\lambda \mathrm{d}t$ gives

$$-\frac{1}{N} = -\lambda t + C.$$

At $t = 0$, $N = N_0$ and the constant $C = -1/N_0$. Hence

$$N = 1/(\lambda t + 1/N_0) = N_0 \frac{1}{(1 + N_0 \lambda t).}$$

Problem 7.4

The integrating factor for the equation

$$L\frac{\mathrm{d}I}{\mathrm{d}t} + RI = V_0$$

is $e^{Rt/L}$. Multiplying both sides by this factor gives

$$\left(\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{R}{L}I\right) e^{Rt/L} = \frac{V_0}{L} e^{Rt/L}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}(Ie^{Rt/L}) = \frac{V_0}{L} e^{Rt/L.}$$

Integrating both sides

$$Ie^{Rt/L} = \frac{V_0}{R} e^{Rt/L} + C$$

and

$$I = \frac{V_0}{R} + Ce^{-Rt/L}.$$

But $I = 0$ for $t = 0$ and hence the constant $C = -V_0/R$ and

$$I = \frac{V_0}{R}(1 - e^{-Rt/L}).$$

Problem 7.5

$$y = \alpha \sin(\omega x + \beta) = \alpha \sin \omega x \cos \beta + \alpha \sin \beta \cos \omega x.$$

Hence

$$A = \alpha \cos \beta \quad \text{and} \quad B = \alpha \sin \beta.$$

Squaring and adding results in

$$\alpha = \sqrt{A^2 + B^2} \quad \text{and} \quad \tan \beta = B/A.$$

Problem 7.6

$$y = \alpha e^{-\gamma x} \sin(\omega x + \beta)$$
$$\mathrm{d}y/\mathrm{d}x = -\alpha\gamma e^{-\gamma x} \sin(\omega x + \beta) + \alpha\omega e^{-\gamma x} \cos(\omega x + \beta)$$

$$\begin{aligned} \mathrm{d}^2/\mathrm{d}x^2 &= \alpha\gamma^2 e^{-\gamma x} \sin(\omega x + \beta)_\alpha \omega\gamma e^{-\gamma x} \cos(\omega x + \beta) \\ &- \alpha\gamma\omega e^{-\gamma x} \cos(\omega x + \beta) - \alpha\omega^2 e^{-\gamma x} \sin(\omega x + \beta) \end{aligned}$$

Substitution in equation 7.7 gives

$$\begin{aligned} &\left[\, a\alpha\gamma^2 e^{-\gamma x} - a\alpha\omega^2 e^{-\gamma x} - b\alpha\gamma e^{-\gamma x} + c\alpha e^{-\gamma x}\right] \sin(\omega x + \beta) \\ &+ \left[-a\alpha\omega\gamma e^{-\gamma x} - a\alpha\omega\gamma e^{-\gamma x} + b\alpha\omega e^{-\gamma x}\right] \cos(\omega x + \beta). \end{aligned}$$

For values of x such that $(\omega x + \beta) = n\pi$ where n is an integer, the cosine term is ± 1 and the sin term is zero. This gives the result

$$-2a\alpha\omega\gamma + \omega b = 0 \text{ and } \gamma = b/2a.$$

For values of x such that $(\omega x + \beta) = (n + 1/2\pi)$ where n is an integer, the sine term is ± 1 and the cos term is zero. This gives

$$a\gamma^2 - a\omega^2 - b\gamma + c = 0 \text{ and } \omega^2 = c/a - b^2/4a^2.$$

Problem 7.7

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \alpha y = x^2.$$

Try as solution

$$y = a + bx + cx^2 + dx^3 + ex^4.$$

Substitution of it and its derivatives into the equation gives

$$2c + 6dx + 12cx^2 + \alpha a + \alpha bx + \alpha cx^2 + \alpha dx^3 + \alpha ex^4 = x^2.$$

Equating the coefficients of the different powers of x on both sides of the above gives $d = e = 0;\ a\alpha + 2c = 0;\ 6d + b\alpha = 0,$ and $12e + c\alpha = 1$. From these relationships, $b = 0;\ c = 1/\alpha$ and $a = -2/\alpha^2$, and the solution to the differential equation is

$$y = -2/\alpha^2 + x^2/\alpha.$$