9 Coordinate systems

Space is three-dimensional, and to define the position of a point requires three numbers, or **coordinates** and a reference system in which to interpret the three numbers. There are three main **coordinate systems** used, each one chosen to most easily describe the situation under study. These three are discussed in this section.

The choice of coordinate system is often dictated by a symmetry exhibited by the problem. For example, looking out from the centre of a cube, all corners are equivalent; looking at a very long straight wire, all points equidistant from the wire are equivalent. Nothing in the description of the cube can distinguish one corner from another, and nothing in the description of effects related to the wire can distinguish one equidistant point from another.

9.1 Cartesian coordinates

The simplest system of coordinates is the **cartesian coordinate system**. In this, three mutually perpendicular axes are drawn through a point chosen to be the origin, the point labelled O on Figure 9.1. The axes are labelled x, y and z, and positive values of these variables are measured along the axes from the origin in the directions shown by the arrows. With this convention the coordinate system shown in the figure is called a **right-handed coordinate system**. The position of a point P is now unambiguously specified by the values of the three **cartesian coordinates** x_P, y_P and z_P. The line PQ on the figure is drawn to be perpendicular to the $x - y$ plane, when the length PQ equals z_P. The lines AQ and BQ are drawn perpendicular to the $x-$ and $y-$ axes respectively, when OA equals x_P and OB is y_P. (The z-coordinate can also be obtained by drawing the perpendicular from P to hit the z axis at a point C when OC equals z_P). Note that each of these coordinates is a scalar.

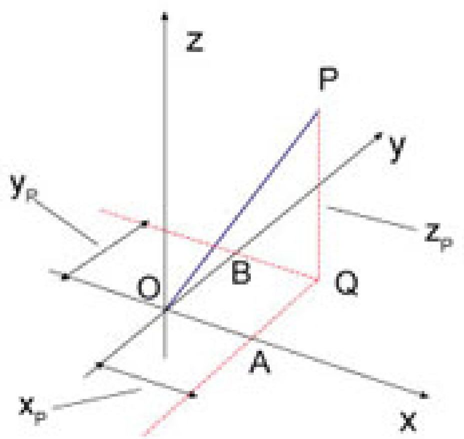

Figure 9.1 Cartesian coordinates.

Problem 9.1 Two small masses, one of 100g the other 200g, are connected by a light rod of length 30 cm which lies in the $x-y$ plane and makes an angle of 30°with the x-axis. Take the centre of mass of the system to be the origin and determine the coordinates of the masses. The rod is displaced upwards by 10 cm, still lying in a plane parallel to the $x-y$ plane, and rotated to make an angle of 60° to the x-axis. Determine the new coordinates.

9.1.1 Cartesian components of vectors

The position of P with respect the origin can also be given in terms of the position vector $\mathbf{r}$, which is a vector in the direction from O to P with magnitude equal to the length of the line OP. The components of the position vector $\mathbf{r}$ of Figure 9.1 along the x, y and z axes are vectors with magnitudes equal to the magnitudes of the cartesian coordinates x, y and z respectively. Defining unit vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ as vectors of unit length along the positive directions of the x, y and z axes respectively, the position vector $\mathbf{r}$ is given in terms of the three cartesian coordinates of P by

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}. \tag{9.1}$$

Descriptions of vectors in terms of their components in the directions of three mutually perpendicular axes are useful in evaluating their additions, subtractions and multiplications. If a vector $\mathbf{a}$ is written as $a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$, and similarly for a vector $\mathbf{b}$, the vector obtained by their addition is given by adding the components in the three directions.

$$(\mathbf{a} + \mathbf{b}) = (a_x + b_x)\mathbf{i} + (a_y + b_y)\mathbf{j} + (a_z + b_z)\mathbf{k}. \tag{9.2}$$

The vector $(\mathbf{a} - \mathbf{b})$ is obtained by subtracting the components.

The scalar product of the two vectors is

$$\mathbf{a}.\mathbf{b} = a_x b_x + a_y b_y + a_z b_z, \tag{9.3}$$

and their vector product is

$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}. \tag{9.4}$$

Example 9.1. Show that the magnitude of the vector $(\mathbf{a} + \mathbf{b})$ is $|(\mathbf{a} + \mathbf{b})| = (a^2 + b^2 + 2ab\cos\theta)^{1/2}$, where θ is the angle between the two vectors.

Solution.

$$(\mathbf{a} + \mathbf{b}).(\mathbf{a} + \mathbf{b}) = a^2 + b^2 + 2\mathbf{a}.\mathbf{b}$$

$$= a^2 + b^2 + 2ab\cos\theta,$$

from equation 1.7. Hence

$$|(\mathbf{a} + \mathbf{b})| = (a^2 + b^2 + 2ab\cos\theta)^{1/2}.$$

9.1.2 Calculations with cartesian coordinates

The cartesian coordinate system is the most straightforward to work with because at any point the directions of the unit vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ remain the same as does the size of an infinitesimal volume element; it is the volume of a small square of sides $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$, namely $\mathrm{d}x\mathrm{d}y\mathrm{d}z$. Descriptions using the cartesian system are conceptually easy but their application in many situations is inconvenient because curved surfaces do not lend themselves to an easy description using coordinates more suited to straight lines and planes.

Example 9.2. Show that the moment of inertia of a square disc of side a when rotated about an axis through its centre perpendicular to the plane of the disc is $(1/6)Ma^2$ where M is the mass of the disc of uniform material.

Solution. The **moment of inertia** I is the sum over the area of the disc of vanishingly small terms like $\rho\ \mathrm{d}x\mathrm{d}y$ times the square of the distance of the element from the axis of rotation, where ρ is the mass per unit area of the material of the disc. A vanishingly small area at position x, y is shown on Figure 9.2. The sum for the moment of inertia is an integral over x and y with the limits on each variable going from $-a/2$ to $+a/2$.

$$I = \rho \int_{-a/2}^{+a/2} \mathrm{d}x \int_{-a/2}^{+a/2} \mathrm{d}y (x^2 + y^2).$$

The integral over x may be done first keeping y constant, when

$$I = \rho \int_{-a/2}^{+a/2} \mathrm{d}y \left(\frac{x^3}{3} + xy^2 \right)_{-a/2}^{+a/2} = \rho \int_{-a/2}^{+a/2} \mathrm{d}y \left(\frac{a^3}{12} + ay^2 \right).$$

Evaluating the definite integral over y and putting the mass of the disc equal to ρa^2 gives the answer above.

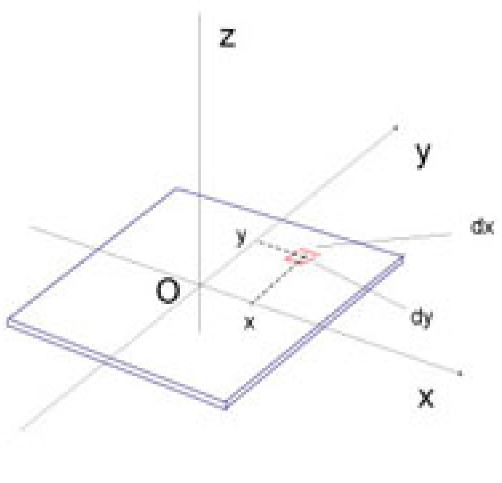

Figure 9.2 The calculation of the moment of inertia of a square disc.

Problem 9.2. Show that the moment of inertia of a solid cube of uniform material for rotation about an axis through its centre and passing through the centres of two opposite faces is equal to $(1/6)Ma^2$, with M the mass of the cube of side a.

9.2 Spherical polar coordinates

Many problems involve forces which depend only on the separation of two objects and point towards their centre of mass, such as gravitational and electrostatic forces. These are called central forces. The coordinate system which is most suited for the discussion of central forces is the **spherical polar coordinate system**.

Figure 9.3 shows the spherical polar coordinates (r, θ, ϕ) of a point P. The coordinate r is the length of the line OP joining the origin to the point. The coordinate θ is the angle the line OP makes with the polar axis, usually called the z-axis, and the coordinate ϕ is the angle between the line OQ and the x-axis. The point Q is where the perpendicular from P meets the $x - y$ plane. In specifying ϕ one starts at a positive point on the x-axis and rotates to meet the line OQ in the sense given by the right-hand screw rule, *viz* that rotation which would advance a right-hand screw in the positive z-direction. The angle ϕ may thus vary from zero to 2π. The angle θ may vary from zero to π.

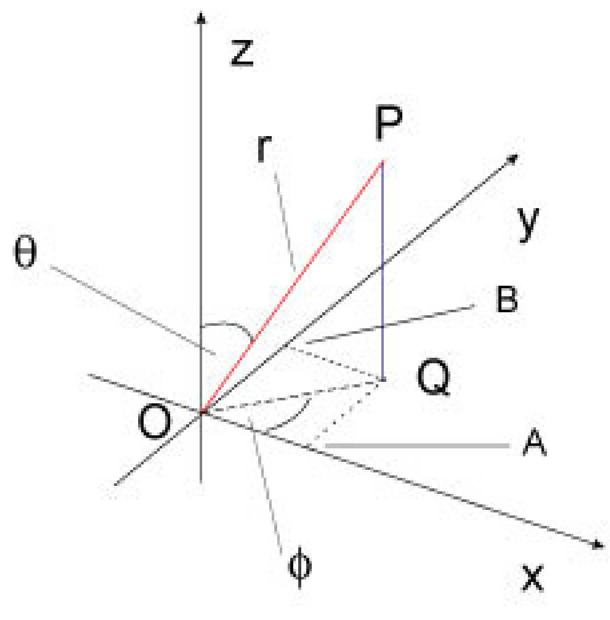

Figure 9.3 Spherical polar coordinates.

Example 9.3. Determine the relationship between the coordinates (x, y, z) of a point and the coordinates (r, θ, ϕ).

Solution. In Figure 9.3, OA=x and OB=y and OQ=$r\sin\theta$. Hence

$$x = r\sin\theta\cos\phi,$$

$$y = r\sin\theta\sin\phi.$$

The length PQ is $r\cos\theta$ and so

$$z = r\cos\theta.$$

9.2.1 Calculations with spherical polar coordinates

The unit vectors in this coordinate system point in different directions as the coordinates change. The unit vector $\hat{\mathbf{r}}$ is in the direction of $\mathbf{r}$ increasing; the unit vector $\hat{\theta}$ in the direction of θ increasing, and the unit vector $\hat{\phi}$ in the direction of ϕ increasing.

The volume $\mathrm{d}V$ of an infinitesimal volume element also differs at different points. Figure 9.4 shows an elementary volume element at a point with coordinates (r, θ, ϕ). One side of the volume has a side of length $\mathrm{d}r$ in the direction of the unit vector $\hat{\mathbf{r}}$, another side of length $r\mathrm{d}\theta$ in the direction of the unit vector $\hat{\theta}$, and a third of length $r\sin\theta\mathrm{d}\phi$ in the direction of the unit vector $\hat{\phi}$, giving

$$\mathrm{d}V = r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi. \qquad 9.5$$

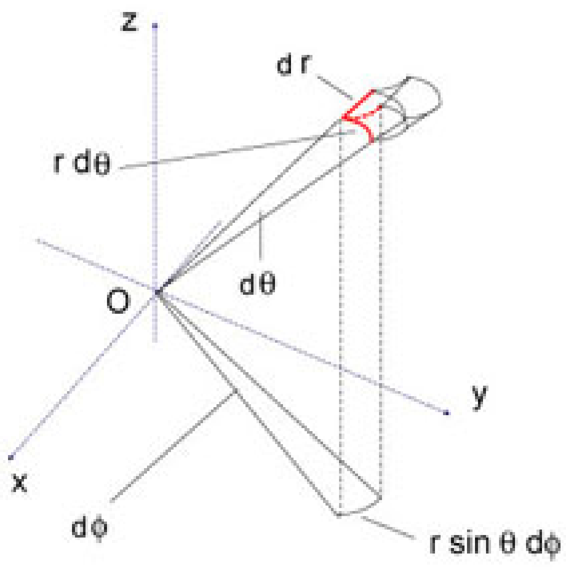

Figure 9.4 An elementary volume element in spherical polar coordinates.

Example 9.4. Show that the surface area of a sphere of radius R is $4\pi R^2$.

Solution. The area of an infinitesimal area element at fixed radius R is $R^2 \ \sin\theta \ \mathrm{d}\theta \ \mathrm{d}\phi$. The surface area A is thus the integral over the complete surface S

$$A = R^2 \int_S \sin\theta \mathrm{d}\theta \mathrm{d}\phi.$$

The limits of the angle θ are from zero to π and those of the angle ϕ from zero to 2π. Hence

$$A = R^2 \int_0^{\pi} \sin\theta \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi.$$

This leads to

$$A = R^2 2\pi (\cos\theta)_{\pi}^{0},$$

and the area is $4\pi R^2$.

Problem 9.3 Show that the volume of a sphere of radius R is $(4/3)\pi R^3$.

Problem 9.4. Show that the volume of an object which is a sphere of radius R with a slice cut off at a distance a from the centre is $(2\pi R^3/3)(1 + a/R)$.

9.3 Cylindrical polar coordinates

An object or physical system often has rotational symmetry; that is if you choose a particular axis through the body or system, rotation about that axis leaves the situation unchanged. There is no way of detecting the rotation. This symmetry can be useful in determining the physical properties of the system. An example is the magnetic field due to a current in a very long straight wire. Things look the same to all observers the same distance from the wire, irrespective of where the wire is viewed from and this leads to the conclusion that the field is the same at all points equidistant from the wire.

In these situations the coordinate system to use to take advantage of such symmetry is the **cylindrical polar coordinate system**. Figure 9.5 shows the cylindrical polar coordinates (r, ϕ, z) of a point P. The coordinate r is the distance AP of the point P from the axis. The coordinate ϕ is the angle the line AP makes with the x-axis, and the coordinate z is the distance OA from the origin to A. In specifying ϕ one starts at a positive point on the x-axis and rotates round to meet the line AP in the sense given by the right-hand screw rule. The angle ϕ may thus vary from zero to 2π.

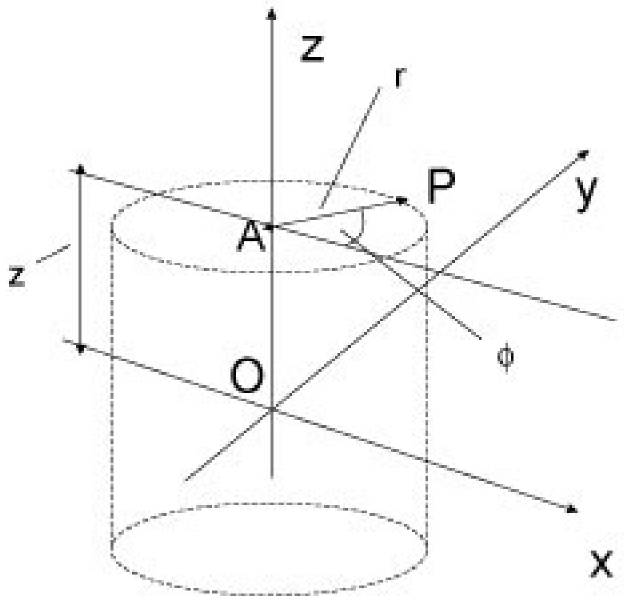

Figure 9.5 Cylindrical polar coordinates.

9.3.1 Calculations with cylindrical polar coordinates

A simple object which has a cylindrical symmetry is a right cylinder or part of a rod with ends cut off straight across. The volume of the bar is given from elementary considerations and is the area of the base, $\pi R^2\times$ the height h, but it is instructive to work it out according to the method one would use for more difficult cases.

Example 9.5, Show that the volume of a cylindrical bar of radius R and height h is $\pi R^2 h$.

Solution. Choose the origin to be at the centre of the cylinder and the $z-$axis to coincide with its axis. Figure 9.6 shows a section cut through the bar perpendicular to the axis at height z. A point in the cylinder is specified by three coordinates r, ϕ and z. Consider an infinitesimal volume element at the point with those coordinates shown in Figure 9.6. The volume of the element is obtained by increasing r by dr, ϕ by dϕ and z by dz. The area of the element in the plane perpendicular to the z-axis is the shaded area shown on th figure and has size d$r \times r$ dϕ. If the thickness of the section is dz, the infinitesimal volume element is d$V = r$ dr dϕ dz. The whole volume is then

$$\int_V \mathrm{d}V = \int_V r\ \mathrm{d}r\ \mathrm{d}\phi\ \mathrm{d}z.$$

The limits of the coordinates over which the volume spreads are r from zero to R; z from $-h/2$ to $+h/2$, and ϕ from zero to 2π and

$$V = \int_{r=0}^{r=R} \int_{z=-h/2}^{z=+h/2} \int_{\phi=0}^{\phi=2pi} r\ \mathrm{d}r\ \mathrm{d}\phi\ \mathrm{d}z.$$

Each integral can now be done independently of the others to give the answer $V = \pi R^2 h$.

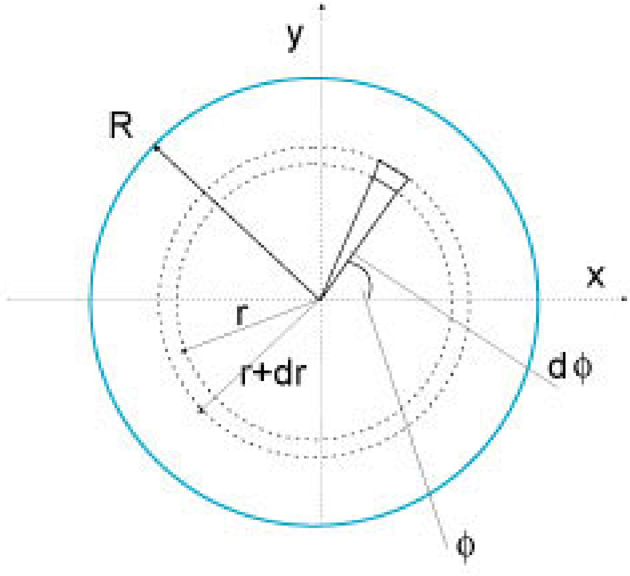

Figure 9.6 An elementary volume element in cylindrical polar coordinates.

Problem 9.5 Show that the volume of an upright cone of height h and base radius R is $\pi R^2 h/3$.

Problem 9.6 A solid cylindrical bar of mass M, height h and radius R is rotated about its axis. Show that the moment of inertia is $\frac{1}{2}MR^2$.
